We start our discussion w/ the maximum principle for symmetric 2-tensors.

Thm. Let $g(t)$ be a family of Riem. metrics on closed M^n. Let $\alpha(t)$ be a family of symmetric 2-tensors satisfying

$$\frac{\partial}{\partial t}\alpha(t) \geq \Delta_{g(t)}\alpha(t) + \langle X(t), \nabla\alpha\rangle + \beta$$

where $\beta(\alpha, g, t)$ is a symmetric 2-tensor which is locally Lipschitz in all of its arguments and satisfies the null eigenvector assumption: whenever $V(x,t)$ is a null eigenvector of $\alpha(t)$, i.e. $(\alpha_{ij}V^j)(x,t) = 0$ we must have

$$\beta(V,V)(x,t) = \left(\beta_{ij}V^iV^j\right)(x,t) \geq 0.$$

If $\alpha(0) \geq 0$ then $\alpha(t) \geq 0$ $\forall$ $t \geq 0$ s.t. the solution exists.

See Hamilton's 3-manifolds paper or Chow–Knopf for a proof of this.

Applications (all in Pset 3)

In dim 3 for closed manifolds:

1) $Ric \geq 0$ is preserved along the RF.

2) The pinching inequality $R_{ij} \geq \varepsilon R g_{ij}$ is preserved along the RF $\forall$ $\varepsilon \leq \frac{1}{3}$.

3) $R_{ij} \leq \frac{1}{2} R g_{ij}$ is preserved along the RF.

4) Let $(M^2, g(t))$ be a RF on a closed surface w/ positive curvature and define

$$Q_{ij} = \nabla_i \nabla_j \log R + \frac{1}{2}\left(R + \frac{1}{t}\right) g_{ij}$$

Then $Q_{ij} \geq 0$ $\forall t$. This is called the matrix Harnack estimate for the RF on surfaces.

Understanding the evolution of Rm

Recall we proved along the RF that

$$\partial_t R_{ijkl} = \Delta R_{ijkl} + \left(R_{ijp}{}^{r} R_{r}{}^{p}{}_{kl} - 2 R_{pik}{}^{r} R_{j}{}^{p}{}_{rl} + 2 R_{pirl} R_{j}{}^{p}{}_{k}{}^{r}\right.$$
$$- \left(R_i{}^p R_{pjkl} + R_j{}^p R_{ipkl} + R_k{}^p R_{ijpl} + R_l{}^p R_{ijkp}\right.$$

On the assignment we prove that if we define a new tensor

$$B_{ijkl} = - R_{ipqj} R_{k}{}^{pq}{}_{l}$$

then $B_{ijkl} = B_{jilk} = B_{klij}$

and

$$\partial_t R_{ijkl} = \Delta R_{ijkl} + 2\left(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}\right)$$
$$- \left(R_i{}^p R_{pjkl} + R_j{}^p R_{ipkl} + R_k{}^p R_{ijpl} + R_l{}^p R_{ijkp}\right).$$

Uhlenbeck's trick

The idea is to fix the initial tangent bundle w/ the initial metric fixed and then to evolve the isometry b/w this fixed bundle and the tangent bundle w/ its evolving metric.

Let $V \to M$ be a v.b. isomorphic to $TM \to M$ w/ fixed time-inde-pendent metric $h = g(0)$. Let

$i_0 : V \to TM$ be the identity map and so $h = i_0^*(g(0))$.

Clearly i_0 is an isometry. Let $g(t)$ be a RF on M w/ $g(0) = g_0$ and we'd like to extend i_0 to a family of bundle isometries i_t so that $h = i_t^*(g(t))$ $\forall$ t. Evolve i by the ODE

$$\frac{\partial}{\partial t} i = \mathrm{Ric} \circ i$$

$$i(\cdot, 0) = i_0$$

where we view Ric as a $(1,1)$-tensor, i.e., an endomorphism of TM using the metric. $\mathrm{Ric}(\partial_i) = R_i^{\ j} \partial_j$.

$\therefore$ we have a 1-parameter family of bundle isomorphism

$i : V \times [0, T) \longrightarrow TM$.

Lemma The bundle map $i_t : (V, h) \longrightarrow (TM, g(t))$ is an isometry $\forall$ t.

Proof :- $\frac{\partial}{\partial t}[g(i_t(v), i_t(v))] = \left(\frac{\partial}{\partial t} g\right)(i(v), i(v)) + 2g(\partial_t i(v), i(v))$

$= -2\mathrm{Ric}(i(v), i(v)) + 2g(\mathrm{Ric}(i(v)), i(v))$

$$= 0$$

$\forall\, v \in V.$

$\Rightarrow$ $i^* g(t)$ is independent of t $\Rightarrow$ $i_t^* g(t) = i_0^* g(0) = h$ $\Rightarrow$

remains an isometry.

As a result, we can look at $i^* Rm$ instead of Rm.

The LC connections $\nabla(t) : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ pull-backs

to connections $D(t)$ on V

$$D(t) = i(t)^* \nabla(t) : \Gamma(TM) \times \Gamma(V) \longrightarrow \Gamma(V) \text{ w/}$$

$$(i^* \nabla)(X, \xi) = (i^* \nabla)_X (\xi) = \nabla_X (\iota(\xi)).$$

And hence we can define $i^* Rm$.

$$i^* Rm(X, Y, Z, W) = Rm(\iota(X), \iota(Y), \iota(Z), \iota(W))$$

for $X, Y, Z, W \in V_x$ for $x \in M$.

Let $\{x^k\}_{k=1}^n$ be local coordinates defined on an open set $\mathcal{U} \subset M^n$

and let $\{e_a\}_{a=1}^n$ be a basis of sections of V restricted to $\mathcal{U}$.

so we can write

$$\iota(e_a) = \sum_{k=1}^n i_a^k \frac{\partial}{\partial x^k}. \quad \text{w/ } i_a^k \text{ the components of the bundle isometry } i(t).$$

$\therefore \; R_{abcd} = (\iota^* Rm)(e_a, e_b, e_c, e_d) = \sum_{i,j,k,l=1}^{n} \iota_a^i \iota_b^j \iota_c^k \iota_d^l R_{ijkl}$

We can define the Laplacian acting on tensor bundles of T^*M and V by

$$\Delta_D = \operatorname{tr}_g (\nabla_D \circ \nabla_D) = \sum_{i,j=1}^{n} g^{ij} (\nabla_D)_i (\nabla_D)_j .$$

w/ $(\nabla_D)_j(\xi) = \nabla_j(\iota(\xi))$.

Lemma If $g(t)$ is a RF and $\iota(t)$ evolves by the ODE described above then $\iota^* Rm$ evolves by

$$\partial_t R_{abcd} = \Delta_D R_{abcd} + 2(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}).$$

w/ $B_{abcd} = h^{eg} h^{fm} R_{aefb} R_{cgmd}$.

Proof. By defn $\partial_t \iota_a^k = R_l^{\ k} \iota_a^l$

$$\Rightarrow \partial_t R_{abcd} = \sum_{i,j,k,l=1}^{n} \partial_t (\iota_a^i \iota_b^j \iota_c^k \iota_d^l R_{ijkl})$$

$$= (\partial_t \iota_a^i) \iota_b^j \iota_c^k \iota_d^l R_{ijkl} + \iota_a^i (\partial_t \iota_b^j) \iota_c^k \iota_d^l R_{ijkl} + \cdot + \cdot + \iota_a^i \iota_b^j \iota_c^k \iota_d^l \partial_t R_{ijkl}$$

$$
\begin{aligned}
&= R^{i}{}_{m}\,\iota^{m}_{a}\,\iota^{j}_{b}\,\iota^{k}_{c}\,\iota^{l}_{d}\,R_{ijkl} + \iota^{i}_{a}\,R^{j}{}_{m}\,\iota^{m}_{b}\,\iota^{k}_{c}\,\iota^{l}_{d}\,R_{ijkl} \\
&\quad + \iota^{i}_{a}\,\iota^{j}_{b}\,R^{k}{}_{m}\,\iota^{m}_{c}\,\iota^{l}_{d}\,R_{ijkl} + \iota^{i}_{a}\,\iota^{j}_{b}\,\iota^{k}_{c}\,R_{m}{}^{l}\,\iota^{m}_{d}\,R_{ijkl} \\
&\quad + \iota^{i}_{a}\,\iota^{j}_{b}\,\iota^{k}_{c}\,\iota^{l}_{d} \Big\{ \Delta R_{ijkl} + 2\left(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}\right) \\
&\qquad - R_{i}{}^{p} R_{pjkl} - R_{j}{}^{p} R_{ipkl} - R_{k}{}^{p} R_{ijpl} - R_{l}{}^{p} R_{ijkp} \Big\}
\end{aligned}
$$

$$
= \iota^{i}_{a}\,\iota^{j}_{b}\,\iota^{k}_{c}\,\iota^{l}_{d} \left\{ \Delta R_{ijkl} + 2\left(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}\right) \right\}
$$

and we get the lemma.

$\blacksquare$

Structure of the evolution equation of Rm

We view Rm as an operator on 2-forms

$$
Rm : \Lambda^2 T^*M \longrightarrow \Lambda^2 T^*M \quad \text{w/}
$$

$$
\left(Rm(U)\right)_{ij} = -R_{ijkl}\,U^{kl}.
$$

If we define the inner product on $\Lambda^2 T^*M$ by

$$
\langle U, V \rangle = U_{ij} V^{ij} \qquad \text{then}
$$

$$\langle Rm(U), V\rangle = R_{ijkl}U^{kl}V^{ij} = R_{klij}V^{ij}U^{kl} = \langle U, Rm(V)\rangle.$$

$\therefore$ we can talk about the square of the curvature operator

$$Rm^2 = Rm \circ Rm : \Lambda^2 T^*M \longrightarrow \Lambda^2 T^*M$$

$$(Rm^2)_{ijkl} = R_{ijps}R^{sp}{}_{kl} = -R_{ijps}R^{ps}{}_{kl}$$

There is another concept of square for curvature which can be described as follows:-

Let $\mathfrak{g}$ be a Lie algebra w/ an inner product $\langle\cdot,\cdot\rangle$. Let $\{\varphi^\alpha\}$ be a basis of $\mathfrak{g}$ and let $C_\gamma^{\alpha\beta}$ be the structure constants

$$[\varphi^\alpha, \varphi^\beta] = \sum_\gamma C_\gamma^{\alpha\beta}\varphi^\gamma .$$

Let $\{\varphi_\alpha^*\}$ be the dual basis s.t. $\varphi_\alpha^*(\varphi^\beta) = \delta_\alpha^\beta$.

So if L is a symmetric bilinear form on $\mathfrak{g}^*$, we regard L as an element of $\mathfrak{g} \otimes_S \mathfrak{g}$ whose components are given by

$$L_{\alpha\beta} = L(\varphi_\alpha^*, \varphi_\beta^*).$$

This gives a commutative bilinear operation $\#$ on symm. bil. forms

L and M by

$$(L \# M)_{\alpha\beta} = C_\alpha^{\gamma\varepsilon} C_\beta^{\delta\zeta} L_{\gamma\delta} M_{\varepsilon\zeta} .$$

$\therefore$ we define the Lie algebra square $L^\# \in \mathfrak{g} \otimes_s \mathfrak{g}$ of L by

$$(L^\#)_{\alpha\beta} = (L \# L)_{\alpha\beta} = C_\alpha^{\gamma\delta} C_\beta^{\varepsilon\zeta} L_{\gamma\varepsilon} L_{\delta\zeta} .$$

Lemma:- If $L \geq 0$ then $L^\# \geq 0$.

Exercise:-

now observe that $\forall x \in M^n$, $\Lambda^2 T_x^* M$ can be given the structure of a Lie algebra $\mathfrak{g} \cong so(n)$ = space of skew-symmetric matrices. Concretely, if $U, V \in \Lambda^2 T^* M$,

$$[U, V]_{ij} = g^{kl} (U_{ik} V_{lj} - V_{ik} U_{lj})$$

In a local o.n. frame $\{e_i\}$ any 2-form $U \cong (U_{ij})$ which is a skew-symmetric matrix so the above formula is just

$$[U, V]_{ij} = (UV - VU)_{ij}$$

and it gives a Lie algebra iso. b/w $\mathfrak{g} = (\Lambda^2 T_x^* M, [\cdot,\cdot]) \cong so(n)$.

Endow this Lie algebra $\mathfrak{g}$ w/ the inner product above and consider the basis elements $\{dx^i \wedge dx^j\}$ w/ the structure constants

$$[dx^p\wedge dx^q, dx^r\wedge dx^s] = \sum_{(ij)} C^{(pq),(rs)}_{(ij)}\, dx^i\wedge dx^j$$

$$C^{(pq),(rs)}_{(ij)} = [\underbrace{dx^p\wedge dx^q}, dx^r\wedge dx^s]_{ij}$$

$$\tfrac{1}{2}(dx^p\otimes dx^q - dx^q\otimes dx^p) = \tfrac{1}{2}(\delta^p_k\delta^q_l - \delta^q_k\delta^p_l)\,dx^k\otimes dx^l$$

$$= \frac{1}{4} g^{kl}\left\{ \begin{aligned} &(\delta^p_i\delta^q_k - \delta^q_i\delta^p_k)(\delta^r_l\delta^s_j - \delta^s_l\delta^r_j) \\ &-(\delta^r_i\delta^s_k - \delta^s_i\delta^r_k)(\delta^p_l\delta^q_j - \delta^q_l\delta^p_j) \end{aligned}\right\}$$

$$= \frac{1}{4}\left\{ \begin{aligned} &g^{qr}(\delta^p_i\delta^s_j - \delta^s_i\delta^p_j) + g^{qs}(\delta^r_i\delta^p_j - \delta^p_i\delta^r_j) \\ &+ g^{pr}(\delta^s_i\delta^q_j - \delta^q_i\delta^s_j) + g^{ps}(\delta^q_i\delta^r_j - \delta^r_i\delta^q_j) \end{aligned}\right\}.$$

$$\therefore\ (Rm^\#)_{ijkl} = R_{pquv} R_{rswx}\, C^{(pq),(rs)}_{(ij)}\, C^{(uv),(wx)}_{(lk)}.$$

The $C^{\cdot\cdot}_{(lk)}$ is NOT a mistake and it shouldn't be $C^{\cdot\cdot}_{(kl)}$. The reason is that by def^n the operator Rm on Λ^2 is defined as $[Rm(U)]_{ij} = -R_{ijkl}U^{kl} = +R_{ijlk}U^{kl}$!

All this is to write our main theorem:-

Theorem If $g(t)$ is a RF then

$$\partial_t(\iota^* Rm) = \Delta_D Rm + Rm^2 + Rm^\#.$$

Proof :- note

$$R_{pquv}\, C^{(pq).(rs)}_{(ij)} = R_{pquv}\frac{1}{4}\left\{ g^{qr}\left(\delta_i^p\delta_j^s - \delta_i^s\delta_j^p\right) + g^{qs}\left(\delta_i^r\delta_j^p - \delta_i^p\delta_j^r\right) + g^{pr}\left(\delta_i^s\delta_j^q - \delta_i^q\delta_j^s\right) + g^{ps}\left(\delta_i^q\delta_j^r - \delta_i^r\delta_j^q\right)\right\}$$

$$= \frac{1}{2} R_{pquv}\left(g^{qr}\left(\delta_i^p\delta_j^s - \delta_i^s\delta_j^p\right) + g^{ps}\left(\delta_i^q\delta_j^r - \delta_i^r\delta_j^q\right)\right)$$

and :

$$(Rm^{\#})_{ijkl} = R_{pquv} R_{rswx}\, C^{(pq)(rs)}_{(ij)}\, C^{(uv)(wx)}_{(lk)}$$

$$= R_{pquv} R_{rswx}\, g^{qr}\left(\delta_i^p\delta_j^s - \delta_i^s\delta_j^p\right) g^{vw}\left(\delta_l^u\delta_k^x - \delta_l^x\delta_k^u\right)$$

$$= R_i{}^r{}_l{}^w R_{rjwk} - R_i{}^r{}_k{}^w R_{rjwl} - R_j{}^r{}_l{}^w R_{riwk} + R_j{}^r{}_k{}^w R_{riwl}$$

$$= -B_{iljk} + B_{ikjl} + B_{jlik} - B_{jkil}$$

$$= 2\left(B_{ikjl} - B_{iljk}\right)$$

and

$$(Rm^2)_{ijkl} = R_{ijps} R^{ps}{}_{lk} = \left(R_{ipsj} + R_{isjp}\right)\left(R_k{}^{sp}{}_l + R_k{}^p{}_l{}^s\right)$$

$$= 2\left(B_{ijkl} - B_{ijlk}\right)$$

and $\therefore$ we get the proof from the previous theorem.

$\blacksquare$